

Mean Field Asymptotic Behavior of Quantum Particles with Initial Correlations

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Abstract. In the paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum large particle systems. One of the developed approaches consists in the description of the evolution of quantum many-particle systems within the framework of marginal observables in mean field scaling limit. Another method based on the possibility to describe the evolution of states within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation in case of initial states specified by a one-particle marginal density operator and correlation operators.

Key words: quantum kinetic equation; quantum Vlasov equation; dual quantum Vlasov hierarchy; mean field scaling limit; correlation operator.

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1 Introduction

As is known the collective behavior of quantum many-particle systems can be effectively described within the framework of a one-particle marginal density operator governed by the kinetic equation in a suitable scaling limit of underlying dynamics. At present the considerable advances in the rigorous derivation of the quantum kinetic equations in the mean (self-consistent) field scaling limit is observed [1]- [6]. In particular, the nonlinear Schrödinger equation [3]- [10] and the Gross–Pitaevskii equation [7]- [15] was justified.

The conventional approach to this problem is based on the consideration of an asymptotic behavior of a solution of the quantum BBGKY hierarchy for marginal density operators constructed within the framework of the theory of perturbations in case of initial data specified by one-particle marginal density operators without correlations, i.e. such that satisfy a chaos condition [16], [17].

In paper [18] it was developed more general method of the derivation of the quantum kinetic equations. By means of a non-perturbative solution of the quantum BBGKY hierarchy constructed in [19] it was established that, if initial data is completely specified by a one-particle marginal density operator, then all possible states of many-particle systems at arbitrary moment of time can

be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation (see also [20]). Then the actual quantum kinetic equations can be derived from the generalized quantum kinetic equation in appropriate scaling limits, for example, in a mean field limit [21].

Another approach to the description of the many-particle evolution is given within the framework of marginal observables governed by the dual quantum BBGKY hierarchy [22]. In paper [23] a rigorous formalism for the description of the kinetic evolution of observables of quantum particles in a mean field scaling limit was developed.

In this paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum particles. Such initial states are typical for the condensed states of quantum gases in contrast to the gaseous state. For example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition specified by correlations of the condensed state [24]. One more example is the influence of initial correlations on ultrafast relaxation processes in plasmas [25], [26].

Thus, our goal consists in the rigorous derivation of the quantum kinetic equations in the presence of initial correlations of quantum large particle systems.

We outline the structure of the paper. In section 2, we establish the mean field asymptotic behavior of marginal observables governed by the dual quantum BBGKY hierarchy. The limit dynamics is described by the set of recurrence evolution equations, namely by the dual quantum Vlasov hierarchy. Furthermore, the links of the dual quantum Vlasov hierarchy for the limit marginal observables and the quantum Vlasov-type kinetic equation with initial correlations are established. In section 3, we consider the relationships of dynamics described by marginal observables and within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation in the presence of initial correlations. In section 4, we develop one more approach to the description of the quantum kinetic evolution with initial correlations in the mean field limit. We prove that a solution of the generalized quantum kinetic equation with initial correlations is governed by the quantum Vlasov-type equation with initial correlations. The property of the propagation of initial correlations is also established. Finally, in section 5, we conclude with some perspectives for future research.

2 The kinetic evolution within the framework of marginal observables

The kinetic evolution of many-particle systems can be described within the framework of observables. We consider this problem on an example of the mean field asymptotic behavior of a non-perturbative solution of the dual quantum BBGKY hierarchy for marginal observables. Moreover, we establish the links of the dual quantum Vlasov hierarchy for the limit marginal observables with the quantum Vlasov-type kinetic equation in the presence of initial correlations.

2.1 Many-particle dynamics of observables

We shall consider a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics in the space \mathbb{R}^3 . We will use units where $\hbar = 2\pi\hbar = 1$ is a Planck constant, and $m = 1$ is the mass of particles.

Let the space \mathcal{H} be a one-particle Hilbert space, then the n -particle space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ is a tensor product of n Hilbert spaces \mathcal{H} . We adopt the usual convention that $\mathcal{H}^{\otimes 0} = \mathbb{C}$. The Fock space over the Hilbert space \mathcal{H} we denote by $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$.

The Hamiltonian H_n of a system of n particles is a self-adjoint operator with the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$

$$H_n = \sum_{i=1}^n K(i) + \epsilon \sum_{i_1 < i_2=1}^n \Phi(i_1, i_2), \quad (1)$$

where $K(i)$ is the operator of a kinetic energy of the i particle, $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential and $\epsilon > 0$ is a scaling parameter. The operator $K(i)$ acts on functions ψ_n , that belong to the subspace $L_0^2(\mathbb{R}^{3n}) \subset \mathcal{D}(H_n) \subset L^2(\mathbb{R}^{3n})$ of infinitely differentiable functions with compact supports, according to the formula: $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$. Correspondingly, we have: $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$, and we assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

Let a sequence $g = (g_0, g_1, \dots, g_n, \dots)$ be an infinite sequence of self-adjoint bounded operators g_n defined on the Fock space $\mathcal{F}_{\mathcal{H}}$. An operator g_n defined on the n -particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ will be also denoted by the symbol $g_n(1, \dots, n)$. Let the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$ be the space of sequences $g = (g_0, g_1, \dots, g_n, \dots)$ of bounded operators g_n defined on the Hilbert space \mathcal{H}_n that satisfy symmetry condition: $g_n(1, \dots, n) = g_n(i_1, \dots, i_n)$, for arbitrary $(i_1, \dots, i_n) \in (1, \dots, n)$, equipped with the operator norm $\|\cdot\|_{\mathfrak{L}(\mathcal{H}_n)}$. We will also consider a more general space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ with the norm

$$\|g\|_{\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})} \doteq \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)},$$

where $0 < \gamma < 1$. We denote by $\mathfrak{L}_{\gamma,0}(\mathcal{F}_{\mathcal{H}}) \subset \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ the everywhere dense set in the space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

For $g_n \in \mathfrak{L}(\mathcal{H}_n)$ it is defined the one-parameter mapping

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_n(t)g_n \doteq e^{itH_n}g_n e^{-itH_n}, \quad (2)$$

where the Hamilton operator H_n has the structure (1). On the space $\mathfrak{L}(\mathcal{H}_n)$ one-parameter mapping (2) is an isometric $*$ -weak continuous group of operators. The infinitesimal generator \mathcal{N}_n of this group of operators is a closed operator for the $*$ -weak topology, and on its domain of the definition $\mathcal{D}(\mathcal{N}_n) \subset \mathfrak{L}(\mathcal{H}_n)$ it is defined in the sense of the $*$ -weak convergence of the space $\mathfrak{L}(\mathcal{H}_n)$ by the operator

$$\text{w}^* - \lim_{t \rightarrow 0} \frac{1}{t} (\mathcal{G}_n(t)g_n - g_n) = -i(g_n H_n - H_n g_n) \doteq \mathcal{N}_n g_n, \quad (3)$$

where H_n is the Hamiltonian (1) and the operator $\mathcal{N}_n g_n$ defined on the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$ has the structure

$$\mathcal{N}_n = \sum_{j=1}^n \mathcal{N}(j) + \epsilon \sum_{j_1 < j_2=1}^n \mathcal{N}_{\text{int}}(j_1, j_2),$$

where

$$\mathcal{N}(j)g_n \doteq -i(g_n K(j) - K(j)g_n), \quad (4)$$

and

$$\mathcal{N}_{\text{int}}(j_1, j_2)g_n \doteq -i(g_n \Phi(j_1, j_2) - \Phi(j_1, j_2)g_n). \quad (5)$$

Therefore on the space $\mathfrak{L}(\mathcal{H}_n)$ a unique solution of the Heisenberg equation for observables of a n -particle system is determined by group (2) [20].

In what follows in this Section we shall hold abridged notations: $Y \equiv (1, \dots, s)$, $X \equiv (j_1, \dots, j_n) \subset Y$, and $\{Y \setminus X\}$ is the set, consisting of a single element $Y \setminus X = (1, \dots, s) \setminus (j_1, \dots, j_n)$, thus, the set $\{Y \setminus X\}$ is a connected subset of the set Y .

To describe the evolution within the framework of marginal observables [22] we introduce a notion of the $(1+n)$ th-order ($n \geq 0$) cumulant of groups of operators (2) as follows [19]

$$\mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \doteq \sum_{P: (\{Y \setminus X\}, X) = \bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} \mathcal{G}_{|\theta(X_i)|}(t, \theta(X_i)), \quad (6)$$

where the symbol \sum_P means the sum over all possible partitions P of the set $(\{Y \setminus X\}, j_1, \dots, j_n)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset (\{Y \setminus X\}, X)$, and $\theta(\cdot)$ is the declusterization mapping defined as follows: $\theta(\{Y \setminus X\}, X) = Y$. For example,

$$\begin{aligned} \mathfrak{A}_1(t, \{Y\}) &= \mathcal{G}_s(t, Y), \\ \mathfrak{A}_2(t, \{Y \setminus (j)\}, j) &= \mathcal{G}_s(t, Y) - \mathcal{G}_{s-1}(t, Y \setminus (j))\mathcal{G}_1(t, j). \end{aligned}$$

In terms of observables the evolution of quantum many-particle systems is described by the sequence $B(t) = (B_0, B_1(t, 1), \dots, B_s(t, 1, \dots, s), \dots)$ of marginal observables (or s -particle observables) $B_s(t, 1, \dots, s)$, $s \geq 1$, determined by the following expansions [22]:

$$B_s(t, Y) = \sum_{n=0}^s \frac{1}{n!} \sum_{j_1 \neq \dots \neq j_n=1}^s \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) B_{s-n}^{0,\epsilon}(Y \setminus X), \quad s \geq 1, \quad (7)$$

where $B(0) = (B_0, B_1^{0,\epsilon}(1), \dots, B_s^{0,\epsilon}(1, \dots, s), \dots) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ is a sequence of initial marginal observables, and the generating operator $\mathfrak{A}_{1+n}(t)$ of expansion (7) is the $(1+n)$ th-order cumulant of groups of operators (2) defined by expansion (6). The simplest examples of marginal observables (7) are given by the expressions:

$$\begin{aligned} B_1(t, 1) &= \mathfrak{A}_1(t, 1)B_1^{0,\epsilon}(1), \\ B_2(t, 1, 2) &= \mathfrak{A}_1(t, \{1, 2\})B_2^{0,\epsilon}(1, 2) + \mathfrak{A}_2(t, 1, 2)(B_1^{0,\epsilon}(1) + B_1^{0,\epsilon}(2)). \end{aligned}$$

If $\gamma < e^{-1}$, for the sequence of operators (7) the following estimate is true

$$\|B(t)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})} \leq e^2(1 - \gamma e)^{-1} \|B(0)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}.$$

We note that a sequence of marginal observables (7) is the non-perturbative solution of recurrence evolution equations known as the dual quantum BBGKY hierarchy [22]:

$$\begin{aligned} \frac{\partial}{\partial t} B_s(t, Y) &= \left(\sum_{j=1}^s \mathcal{N}(j) + \sum_{j_1 < j_2 = 1}^s \mathcal{N}_{\text{int}}(j_1, j_2) \right) B_s(t, Y) + \\ &+ \sum_{j_1 \neq j_2 = 1}^s \mathcal{N}_{\text{int}}(j_1, j_2) B_{s-1}(t, Y \setminus (j_1)), \\ B_s(t)|_{t=0} &= B_s^0, \quad s \geq 1. \end{aligned}$$

We adduce also the relationship of marginal observables governed by hierarchy (7) and observables governed the Heisenberg equations [20]

$$B_s(t, Y) \doteq \sum_{n=0}^s \frac{(-1)^n}{n!} \sum_{j_1 \neq \dots \neq j_n = 1}^s (\mathcal{G}_{s-n}(t) A_{s-n}^0)(Y \setminus (j_1, \dots, j_n)), \quad s \geq 1,$$

where the group $\mathcal{G}_{s-n}(t)$ is defined by formula (2) and the operators A_{s-n}^0 , $0 \leq n \leq s$, are initial observables.

2.2 A mean field asymptotic behavior of marginal observables

A mean field asymptotic behavior of marginal observables (7) is described by the following theorem [23].

Theorem 1. *Let for $B_n^{0,\epsilon} \in \mathfrak{L}(\mathcal{H}_n)$, $n \geq 1$, in the sense of the $*$ -weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$ it holds: $w^* - \lim_{\epsilon \rightarrow 0} (\epsilon^{-n} B_n^{0,\epsilon} - b_n^0) = 0$, then for arbitrary finite time interval there exists the mean field limit of marginal observables (7): $w^* - \lim_{\epsilon \rightarrow 0} (\epsilon^{-s} B_s(t) - b_s(t)) = 0$, $s \geq 1$, that are determined by the following expansions:*

$$\begin{aligned} b_s(t, Y) &= \sum_{n=0}^{s-1} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1 = 1}^s \mathcal{N}_{\text{int}}(i_1, j_1) \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \dots \quad (8) \\ &\prod_{l_n \in Y \setminus (j_1, \dots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n) \sum_{\substack{i_n \neq j_n = 1, \\ i_n, j_n \neq (j_1, \dots, j_{n-1})}}^s \mathcal{N}_{\text{int}}(i_n, j_n) \\ &\times \prod_{l_{n+1} \in Y \setminus (j_1, \dots, j_n)} \mathcal{G}_1(t_n, l_{n+1}) b_{s-n}^0(Y \setminus (j_1, \dots, j_n)), \end{aligned}$$

where the operator $\mathcal{N}_{\text{int}}(i_1, j_2)$ is defined on $g_n \in \mathfrak{L}(\mathcal{H}_n)$ by formula (5).

The proof of Theorem 1 is based on formulas for cumulants of asymptotically perturbed groups of operators (2).

For arbitrary finite time interval the asymptotically perturbed group of operators (2) has the following scaling limit in the sense of the $*$ -weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$:

$$w^* - \lim_{\epsilon \rightarrow 0} \left(\mathcal{G}_s(t, Y) - \prod_{j=1}^s \mathcal{G}_1(t, j) \right) g_s = 0. \quad (9)$$

Taking into account analogs of the Duhamel equations for cumulants of asymptotically perturbed groups of operators, in view of formula (9) we have

$$\begin{aligned} & \text{w}^* - \lim_{\epsilon \rightarrow 0} \left(\epsilon^{-n} \frac{1}{n!} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, j_1, \dots, j_n) - \right. \\ & - \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1=1}^s \mathcal{N}_{\text{int}}(i_1, j_1) \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \dots \\ & \left. \prod_{l_n \in Y \setminus (j_1, \dots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n) \sum_{\substack{i_n \neq j_n=1, \\ i_n, j_n \neq (j_1, \dots, j_{n-1})}}^s \mathcal{N}_{\text{int}}(i_n, j_n) \prod_{l_{n+1} \in Y \setminus (j_1, \dots, j_n)} \mathcal{G}_1(t_n, l_{n+1}) \right) g_{s-n} = 0, \end{aligned}$$

where we used notations accepted in formula (8) and $g_{s-n} \equiv g_{s-n}((1, \dots, s) \setminus (j_1, \dots, j_n))$, $n \geq 1$. As a result of this equality we establish the validity of Theorem 1 for expansion (7) of marginal observables.

If $b^0 \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, then the sequence $b(t) = (b_0, b_1(t), \dots, b_s(t), \dots)$ of limit marginal observables (8) is a generalized global solution of the Cauchy problem of the dual quantum Vlasov hierarchy

$$\frac{\partial}{\partial t} b_s(t, Y) = \sum_{j=1}^s \mathcal{N}(j) b_s(t, Y) + \sum_{j_1 \neq j_2=1}^s \mathcal{N}_{\text{int}}(j_1, j_2) b_{s-1}(t, Y \setminus (j_1)), \quad (10)$$

$$b_s(t)|_{t=0} = b_s^0, \quad s \geq 1, \quad (11)$$

where the infinitesimal generator $\mathcal{N}(j)$ of the group of operators $\mathcal{G}_1(t, j)$ of j particle is defined on $g_n \in \mathfrak{L}_0(\mathcal{H}_n)$ by formula (4).

It should be noted that equations set (10) has the structure of recurrence evolution equations. We give several examples of the evolution equations of the dual quantum Vlasov hierarchy (10) in terms of operator kernels of the limit marginal observables

$$\begin{aligned} i \frac{\partial}{\partial t} b_1(t, q_1; q'_1) &= -\frac{1}{2} (-\Delta_{q_1} + \Delta_{q'_1}) b_1(t, q_1; q'_1), \\ i \frac{\partial}{\partial t} b_2(t, q_1, q_2; q'_1, q'_2) &= -\frac{1}{2} \sum_{i=1}^2 (-\Delta_{q_i} + \Delta_{q'_i}) b_2(t, q_1, q_2; q'_1, q'_2) + \\ &+ (\Phi(q'_1 - q'_2) - \Phi(q_1 - q_2)) (b_1(t, q_1; q'_1) + b_1(t, q_2; q'_2)). \end{aligned}$$

We consider the mean field limit of a particular case of marginal observables, namely the additive-type marginal observables $B^{(1)}(0) = (0, B_1^{0,\epsilon}(1), 0, \dots)$. We remark that the k -ary marginal observables are represented by the sequence $B^{(k)}(0) = (0, \dots, 0, B_k^{0,\epsilon}(1, \dots, k), 0, \dots)$. In case of additive-type marginal observables expansions (7) take the following form:

$$B_s^{(1)}(t, Y) = \mathfrak{A}_s(t) \sum_{j=1}^s B_1^{0,\epsilon}(j), \quad s \geq 1, \quad (12)$$

where the operator $\mathfrak{A}_s(t)$ is s -order cumulant (6) of groups of operators (2).

Corollary 1. *If for the additive-type marginal observable $B_1^{0,\epsilon} \in \mathfrak{L}(\mathcal{H})$ it holds $w^* - \lim_{\epsilon \rightarrow 0} (\epsilon^{-1} B_1^{0,\epsilon} - b_1^0) = 0$, then, according to statement of Theorem 1, for additive-type marginal observable (12) we have: $w^* - \lim_{\epsilon \rightarrow 0} (\epsilon^{-s} B_s^{(1),\epsilon}(t) - b_s^{(1)}(t)) = 0$, $s \geq 1$, where the limit additive-type marginal observable $b_s^{(1)}(t)$ is determined by a special case of expansion (8)*

$$\begin{aligned}
b_s^{(1)}(t, Y) &= \int_0^t dt_1 \dots \int_0^{t_{s-2}} dt_{s-1} \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1=1}^s \mathcal{N}_{\text{int}}(i_1, j_1) \\
&\times \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \dots \prod_{l_{s-1} \in Y \setminus (j_1, \dots, j_{s-2})} \mathcal{G}_1(t_{s-2} - t_{s-1}, l_{s-1}) \\
&\times \sum_{\substack{i_{s-1} \neq j_{s-1}=1, \\ i_{s-1}, j_{s-1} \neq (j_1, \dots, j_{s-2})}}^s \mathcal{N}_{\text{int}}(i_{s-1}, j_{s-1}) \prod_{l_s \in Y \setminus (j_1, \dots, j_{s-1})} \mathcal{G}_1(t_{s-1}, l_s) b_1^0(Y \setminus (j_1, \dots, j_{s-1})).
\end{aligned} \tag{13}$$

We make several examples of expansions (13) for the limit additive-type marginal observables

$$\begin{aligned}
b_1^{(1)}(t, 1) &= \mathcal{G}_1(t, 1) b_1^0(1), \\
b_2^{(1)}(t, 1, 2) &= \int_0^t dt_1 \prod_{i=1}^2 \mathcal{G}_1(t - t_1, i) \mathcal{N}_{\text{int}}(1, 2) \sum_{j=1}^2 \mathcal{G}_1(t_1, j) b_1^0(j).
\end{aligned}$$

Thus, for arbitrary initial states in the mean field scaling limit the kinetic evolution of quantum many-particle systems is described in terms of limit marginal observables (8) governed by the dual quantum Vlasov hierarchy (10).

2.3 The derivation of the quantum Vlasov-type kinetic equation with initial correlations

Furthermore, the relationships between the evolution of observables and the kinetic evolution of states described in terms of a one-particle marginal density operator are discussed.

We shall consider initial states of a quantum many-particle system specified by the one-particle (marginal) density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ in the presence of correlations, i.e. initial state specified by the following sequence of marginal density operators

$$F^c = (1, F_1^{0,\epsilon}(1), g_2^\epsilon \prod_{i=1}^2 F_1^{0,\epsilon}(i), \dots, g_n^\epsilon \prod_{i=1}^n F_1^{0,\epsilon}(i), \dots), \tag{14}$$

where the bounded operators $g_n^\epsilon \equiv g_n^\epsilon(1, \dots, n) \in \mathfrak{L}(\mathcal{H}_n)$, $n \geq 2$, are specified the initial correlations. We remark that such assumption about initial states is intrinsic for the kinetic description of a gas. On the other hand, initial data (14) is typical for the condensed states of quantum gases, for example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition with the correlations which characterize the condensed state [24].

We assume that for the initial one-particle (marginal) density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ exists the mean field limit $\lim_{\epsilon \rightarrow 0} \|\epsilon F_1^{0,\epsilon} - f_1^0\|_{\mathfrak{L}^1(\mathcal{H})} = 0$, and it holds: $\lim_{\epsilon \rightarrow 0} \|g_n^\epsilon - g_n\|_{\mathfrak{L}^1(\mathcal{H}_n)} = 0$, then in

the mean field limit initial state is specified by the following sequence of limit operators

$$f^c = (1, f_1^0(1), g_2 \prod_{i=1}^2 f_1^0(i), \dots, g_n \prod_{i=1}^n f_1^0(i), \dots). \quad (15)$$

We note that in case of initial states specified by sequence (15) the average values (mean values) of limit marginal observables (8) are determined by the following positive continuous linear functional [20]

$$(b(t), f^c) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{1,\dots,n} b_n(t, 1, \dots, n) g_n(1, \dots, n) \prod_{i=1}^n f_1^0(i). \quad (16)$$

For $b(t) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $f_1^0 \in \mathfrak{L}^1(\mathcal{H})$, functional (16) exists under the condition that $\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})} < \gamma$.

We consider relationships of the constructed mean field asymptotic behavior of marginal observables with the quantum Vlasov-type kinetic equation in case of initial states (15).

For the limit additive-type marginal observables (13) the following equality is true

$$\begin{aligned} (b^{(1)}(t), f^c) &= \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\dots,s} b_s^{(1)}(t, 1, \dots, s) g_s(1, \dots, s) \prod_{i=1}^s f_1^0(i) = \\ &= \text{Tr}_1 b_1^0(1) f_1(t, 1), \end{aligned}$$

where the operator $b_s^{(1)}(t)$ is determined by expansion (13) and the one-particle (marginal) density operator $f_1(t, 1)$ is represented by the series expansion

$$\begin{aligned} f_1(t, 1) &= \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \text{Tr}_{2,\dots,n+1} \mathcal{G}_1^*(t - t_1, 1) \mathcal{N}_{\text{int}}^*(1, 2) \prod_{j_1=1}^2 \mathcal{G}_1^*(t_1 - t_2, j_1) \dots \\ &\times \prod_{i_n=1}^n \mathcal{G}_1^*(t_n - t_n, i_n) \sum_{k_n=1}^n \mathcal{N}_{\text{int}}^*(k_n, n+1) \prod_{j_n=1}^{n+1} \mathcal{G}_1^*(t_n, j_n) g_{1+n}(1, \dots, n+1) \prod_{i=1}^{n+1} f_1^0(i). \end{aligned} \quad (17)$$

In series (17) the operator $\mathcal{N}_{\text{int}}^*(j_1, j_2) f_n = -\mathcal{N}_{\text{int}}(j_1, j_2) f_n$ is an adjoint operator to operator (3) and the group $\mathcal{G}_1^*(t, i) = \mathcal{G}_1(-t, i)$ is dual to group (2) in the sense of functional (16). For bounded interaction potentials series (17) is norm convergent on the space $\mathfrak{L}^1(\mathcal{H})$ under the condition that $t < t_0 \equiv (2 \|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)} \|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$.

The operator $f_1(t)$ represented by series (17) is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations:

$$\frac{\partial}{\partial t} f_1(t, 1) = \mathcal{N}^*(1) f_1(t, 1) + \quad (18)$$

$$+ \text{Tr}_2 \mathcal{N}_{\text{int}}^*(1, 2) \prod_{i_1=1}^2 \mathcal{G}_1^*(t, i_1) g_2(1, 2) \prod_{i_2=1}^2 (\mathcal{G}_1^*)^{-1}(t, i_2) f_1(t, 1) f_1(t, 2),$$

$$f_1(t)|_{t=0} = f_1^0, \quad (19)$$

where the operator $\mathcal{N}^*(1) = -\mathcal{N}(1)$ is an adjoint operator to operator (4) in the sense of functional (16) and the group $(\mathcal{G}_1^*)^{-1}(t) = \mathcal{G}_1^*(-t) = \mathcal{G}_1(t)$ is inverse to the group $(\mathcal{G}_1^*)(t)$. This fact is proved similarly as in case of a solution of the quantum BBGKY hierarchy represented by the iteration series [20] (see also [27], [28]).

Thus, in case of initial states specified by one-particle (marginal) density operator (15) we establish that the dual quantum Vlasov hierarchy (10) for additive-type marginal observables describes the evolution of quantum large particle system just as the quantum Vlasov-type kinetic equation with initial correlations (18).

2.4 The mean field evolution of initial correlations

The property of the propagation of initial correlations is a consequence of the validity of the following equality for the mean value functionals of the limit k -ary marginal observables in case of $k \geq 2$

$$\begin{aligned} (b^{(k)}(t), f^c) &= \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\dots,s} b_s^{(k)}(t, 1, \dots, s) g_s(1, \dots, s) \prod_{j=1}^s f_1^0(j) = \\ &= \frac{1}{k!} \text{Tr}_{1,\dots,k} b_k^0(1, \dots, k) \prod_{i_1=1}^k \mathcal{G}_1^*(t, i_1) g_k(1, \dots, k) \prod_{i_2=1}^k \mathcal{G}_1^*(-t, i_2) \prod_{j=1}^k f_1(t, j), \quad k \geq 2, \end{aligned} \quad (20)$$

where the limit one-particle (marginal) density operator $f_1(t, j)$ is represented by series expansion (17) and therefore it is governed by the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (18), (19).

This fact is proved similarly to the proof of a property on the propagation of initial chaos in a mean field scaling limit [23].

Thus, in case of the limit k -ary marginal observables a solution of the dual quantum Vlasov hierarchy (10) is equivalent to a property of the propagation of initial correlations for the k -particle marginal density operator in the sense of equality (20) or in other words the mean field scaling dynamics does not create correlations.

We remark that the general approaches to the description of the evolution of states of quantum many-particle systems within the framework of correlation operators and marginal correlation operators were given in papers [29], [30] and [31], respectively (see also review [20]).

3 On relationships of dynamics of observables and kinetic evolution of states

We consider the relationships of dynamics of quantum many-particle systems described in terms of marginal observables and dynamics described within the framework of a one-particle (marginal) density operator governed by the quantum kinetic equation in the presence of initial correlations in the general case, i.e. without any approximations like scaling limits as above in Section 2. If initial states is completely specified by a one-particle (marginal) density operator, using a non-perturbative solution of the dual quantum BBGKY hierarchy we prove that all possible states at arbitrary moment of time can be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation with initial correlations.

3.1 Quantum dynamics of initial states specified by the one-particle density operator and correlations

In case of initial states specified by sequence (14) the average values (mean values) of marginal observables (7) are defined by the positive continuous linear functional on the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$

$$(B(t), F^c) \doteq \sum_{s=0}^{\infty} \frac{1}{s!} \text{Tr}_{1,\dots,s} B_s(t, 1, \dots, s) g_s^\epsilon(1, \dots, s) \prod_{i=1}^s F_1^{0,\epsilon}(i). \quad (21)$$

For $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ and $B_s^{0,\epsilon} \in \mathfrak{L}(\mathcal{H}_s)$ series (21) exists under the condition that $\|F_1^{0,\epsilon}\|_{\mathfrak{L}^1(\mathcal{H})} < e^{-1}$.

For mean value functional (21) the following representation holds

$$(B(t), F^c) = (B(0), F(t | F_1(t))), \quad (22)$$

where $B(0) = (B_0, B_1^{0,\epsilon}(1), \dots, B_s^{0,\epsilon}(1, \dots, s), \dots) \in \mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ is a sequence of initial marginal observables, and $F(t | F_1(t)) = (1, F_1(t), F_2(t | F_1(t)), \dots, F_s(t | F_1(t)), \dots)$ is a sequence of explicitly defined marginal functionals $F_s(t | F_1(t))$, $s \geq 2$, with respect to the following one-particle (marginal) density operator

$$F_1(t, 1) = \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{2,\dots,1+n} \mathfrak{A}_{1+n}^*(t) g_{n+1}^\epsilon(1, \dots, n+1) \prod_{i=1}^{n+1} F_1^{0,\epsilon}(i). \quad (23)$$

The generating operator $\mathfrak{A}_{1+n}^*(t) \equiv \mathfrak{A}_{1+n}^*(t, 1, \dots, n+1)$ of series expansion (23) is the $(1+n)th$ -order cumulant of groups of operators $\mathcal{G}_n^*(t)$, $n \geq 1$, dual to groups (2) in the sense of functional (21), namely

$$\mathfrak{A}_{1+n}^*(t, 1, \dots, n+1) \doteq \sum_{P: (1,\dots,n+1)=\bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} \mathcal{G}_{|X_i|}^*(t, X_i),$$

where the symbol \sum_P means the sum over all possible partitions P of the set $(1, \dots, n+1)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset (1, \dots, n+1)$.

The marginal functionals of the state $F_s(t | F_1(t))$, $s \geq 2$, are represented by the following series expansions:

$$F_s(t, Y | F_1(t)) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \text{Tr}_{s+1,\dots,s+n} \mathfrak{G}_{1+n}(t, \{Y\}, X \setminus Y) \prod_{i=1}^{s+n} F_1(t, i), \quad (24)$$

where we denote: $Y \equiv (1, \dots, s)$, $X \setminus Y \equiv (s+1, \dots, s+n)$, and the $(1+n)th$ -order generating operator $\mathfrak{G}_{1+n}(t)$, $n \geq 0$, of this series is determined by the following expansion

$$\begin{aligned} \mathfrak{G}_{1+n}(t, \{Y\}, X \setminus Y) &\doteq n! \sum_{k=0}^n (-1)^k \sum_{n_1=1}^n \dots \sum_{n_k=1}^{n-n_1-\dots-n_{k-1}} \frac{1}{(n - n_1 - \dots - n_k)!} \\ &\times \check{\mathfrak{A}}_{1+n-n_1-\dots-n_k}(t, \{Y\}, s+1, \dots, s+n-n_1-\dots-n_k) \\ &\times \prod_{j=1}^k \sum_{\substack{D_j: Z_j = \bigcup_{l_j} X_{l_j}, \\ |D_j| \leq s+n-n_1-\dots-n_j}} \frac{1}{|D_j|!} \sum_{i_1 \neq \dots \neq i_{|D_j|=1}}^{s+n-n_1-\dots-n_j} \prod_{X_{l_j} \subset D_j} \frac{1}{|X_{l_j}|!} \check{\mathfrak{A}}_{1+|X_{l_j}|}(t, i_{l_j}, X_{l_j}). \end{aligned} \quad (25)$$

In formula (25) we denote by $\sum_{D_j: Z_j = \cup_{i_j} X_{i_j}}$ the sum over all possible dissections of the linearly ordered set $Z_j \equiv (s + n - n_1 - \dots - n_j + 1, \dots, s + n - n_1 - \dots - n_{j-1})$ on no more than $s + n - n_1 - \dots - n_j$ linearly ordered subsets and we introduced the $(1 + n)th$ -order scattering cumulants

$$\check{\mathfrak{A}}_{1+n}(t, \{Y\}, X \setminus Y) \doteq \mathfrak{A}_{1+n}^*(t, \{Y\}, X \setminus Y) g_{s+n}^\epsilon(\theta(\{Y\}), X \setminus Y) \prod_{i=1}^{s+n} (\mathfrak{A}_1^*)^{-1}(t, i),$$

where the operator $g_{s+n}^\epsilon(\theta(\{Y\}), X \setminus Y)$ is specified initial correlations (15), the operator $(\mathfrak{A}_1^*)^{-1}(t)$ is inverse to the operator $\mathfrak{A}_1^*(t)$ and it is used notations accepted above. We give examples of the scattering cumulants

$$\begin{aligned} \mathfrak{G}_1(t, \{Y\}) &= \check{\mathfrak{A}}_1(t, \{Y\}) \doteq \mathfrak{A}_1^*(t, \{Y\}) g_s^\epsilon(\theta(\{Y\})) \prod_{i=1}^s (\mathfrak{A}_1^*)^{-1}(t, i), \\ \mathfrak{G}_2(t, \{Y\}, s+1) &= \mathfrak{A}_2^*(t, \{Y\}, s+1) g_{s+1}^\epsilon(\theta(\{Y\}), s+1) \prod_{i=1}^{s+1} (\mathfrak{A}_1^*)^{-1}(t, i) - \\ &\quad - \mathfrak{A}_1^*(t, \{Y\}) g_s^\epsilon(\theta(\{Y\})) \prod_{i=1}^s (\mathfrak{A}_1^*)^{-1}(t, i) \sum_{i=1}^s \mathfrak{A}_2^*(t, i, s+1) g_2^\epsilon(i, s+1) (\mathfrak{A}_1^*)^{-1}(t, i) (\mathfrak{A}_1^*)^{-1}(t, s+1). \end{aligned}$$

If $\|F_1(t)\|_{\mathfrak{L}^1(\mathcal{H})} < e^{-(3s+2)}$, then for arbitrary $t \in \mathbb{R}$ series expansion (22) converges in the norm of the space $\mathfrak{L}^1(\mathcal{H}_s)$ [20].

We emphasize that marginal functionals of the state (24) characterize the correlations generated by dynamics of quantum many-particle systems in the presence of initial correlations.

3.2 On an equivalence of mean value functional representations

We prove the validity of equality (22) for mean value functional (21).

In a particular case of initial data specified by the additive-type marginal observables, i.e. $B^{(1)}(0) = (0, B_1^{0,\epsilon}(1), 0, \dots)$, equality (22) takes the form

$$(B^{(1)}(t), F^c) = \text{Tr}_1 B_1^{0,\epsilon}(1) F_1(t, 1), \quad (26)$$

where the one-particle (marginal) density operator $F_1(t)$ is determined by series expansion (23). The validity of this equality is a result of the direct transformation of the generating operators of expansions (12) to adjoint operators in the sense of the functional (21).

In case of initial data specified by the s -ary marginal observables i.e. $B^{(s)}(0) = (0, \dots, 0, B_s^{0,\epsilon}(1, \dots, s), 0, \dots)$, $s \geq 2$, equality (22) takes the following form:

$$(B^{(s)}(t), F^c) = \frac{1}{s!} \text{Tr}_{1,\dots,s} B_s^{0,\epsilon}(1, \dots, s) F_s(t, 1, \dots, s | F_1(t)), \quad (27)$$

where the marginal functional of the state $F_s(t | F_1(t))$ is represented by series expansion (24).

The proof of equality (27) is based on the application of cluster expansions to generating operators (6) of expansions (7) which is dual to the kinetic cluster expansions introduced in paper [18]. Then the adjoint series expansion can be expressed in terms of one-particle (marginal) density operator (23) in the form of the functional from the right-hand side of equality (27).

In case of the general type of marginal observables the validity of equality (22) is proven in much the same way as the validity of particular equalities (26) and (27).

3.3 The generalized quantum kinetic equation with initial correlations

As a result of the differentiation over the time variable of operator represented by series (23) in the sense of the norm convergence of the space $\mathfrak{L}^1(\mathcal{H})$, then the application of the kinetic cluster expansions [18], [32] to the generating operators of obtained series expansion, for the one-particle (marginal) density operator we derive the following identity

$$\frac{\partial}{\partial t} F_1(t, 1) = \mathcal{N}^*(1) F_1(t, 1) + \epsilon \text{Tr}_2 \mathcal{N}_{\text{int}}^*(1, 2) F_2(t, 1, 2 | F_1(t)), \quad (28)$$

where the operators $\mathcal{N}^*(1) = -\mathcal{N}(1)$ and $\mathcal{N}_{\text{int}}^*(1, 2) = -\mathcal{N}_{\text{int}}(1, 2)$ are adjoint operators in the sense of functional (16) to operators (4) and (5), respectively, and the collision integral is determined by series expansion (24) for the marginal functional of the state in case of $s = 2$. This identity we treat as the non-Markovian quantum kinetic equation. We refer to this evolution equation as the generalized quantum kinetic equation with initial correlations.

We emphasize that the coefficients in an expansion of the collision integral of kinetic equation (28) are determined by the operators specified initial correlations (14). We remark also that in case of a system of particles with a n -body interaction potential the collision integral of the corresponding quantum kinetic equation is determined by the marginal functional of the state (24) in case of $s = n$ [20].

For the generalized quantum kinetic equation with initial correlations (28) on the space $\mathfrak{L}^1(\mathcal{H})$ the following statement is true.

If $\|F_1^{0,\epsilon}\|_{\mathfrak{L}^1(\mathcal{H})} < (e(1 + e^9))^{-1}$, the global in time solution of initial-value problem of kinetic equation (28) is determined by series expansion (23). For initial data $F_1^{0,\epsilon} \in \mathfrak{L}_0^1(\mathcal{H})$ it is a strong (classical) solution and for an arbitrary initial data it is a weak (generalized) solution.

We note that for initial data (14) specified by a one-particle (marginal) density operator, the evolution of states described within the framework of a one-particle (marginal) density operator governed by the generalized quantum kinetic equation with initial correlations (28) is dual to the dual quantum BBGKY hierarchy for additive-type marginal observables with respect to bilinear form (21), and it is completely equivalent to the description of states in terms of marginal density operators governed by the quantum BBGKY hierarchy.

Thus, the evolution of quantum many-particle systems described in terms of marginal observables can be also described within the framework of a one-particle (marginal) density operator governed by the generalized quantum kinetic equation with initial correlations (28).

4 The asymptotic behavior of the generalized quantum kinetic equation with initial correlations

We construct a mean field asymptotics of a solution of the generalized quantum kinetic equation with initial correlations (28). This asymptotics is governed by the quantum Vlasov-type kinetic equation with initial correlations (18) derived above from the dual quantum Vlasov hierarchy (10) for the limit marginal observables. Moreover, a mean field asymptotic behavior of marginal functionals of the state (24) describes the propagation in time of initial correlations like established property (20).

4.1 The limit theorem

For solution (23) of the generalized quantum kinetic equation with initial correlations (28) the following mean field limit theorem is true [32].

Theorem 2. *If for the initial one-particle density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ exists the following limit: $\lim_{\epsilon \rightarrow 0} \|\epsilon F_1^{0,\epsilon} - f_1^0\|_{\mathfrak{L}^1(\mathcal{H})} = 0$ and $\lim_{\epsilon \rightarrow 0} \|g_n^\epsilon - g_n\|_{\mathfrak{L}^1(\mathcal{H}_n)} = 0$, then for $t \in (-t_0, t_0)$, where $t_0 \equiv (2 \|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)} \|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$, there exists the mean field limit of solution (23) of the Cauchy problem of the generalized quantum kinetic equation with initial correlations (28)*

$$\lim_{\epsilon \rightarrow 0} \|\epsilon F_1(t) - f_1(t)\|_{\mathfrak{L}^1(\mathcal{H})} = 0, \quad (29)$$

where the operator $f_1(t)$ is represented by series expansion (17) and it is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (18),(19).

The proof of this theorem is based on formulas of asymptotically perturbed cumulants of groups of operators $\mathcal{G}_n^*(t)$, $n \geq 1$, adjoint to groups (2) in the sense of functional (21). Indeed, in a mean field limit for generating evolution operators (25) of series expansion (24) the following equalities are valid:

$$\lim_{\epsilon \rightarrow 0} \left\| \frac{1}{\epsilon^n} \mathfrak{G}_{1+n}(t, \{Y\}, X \setminus Y) f_{s+n} \right\|_{\mathfrak{L}^1(\mathcal{H}_{s+n})} = 0, \quad n \geq 1, \quad (30)$$

and in case of the first-order generating evolution operator we have

$$\lim_{\epsilon \rightarrow 0} \left\| \left(\mathfrak{G}_1(t, \{Y\}) - \prod_{j_1=1}^s \mathcal{G}_1^*(t, j_1) g_s(1, \dots, s) \prod_{j_2=1}^s \mathcal{G}_1^*(-t, j_2) \right) f_s \right\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0, \quad (31)$$

respectively.

In view that under the condition $t < t_0 \equiv (2 \|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)} \|\epsilon F_1^{0,\epsilon}\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$, for a bounded interaction potential the series for the operator $\epsilon F_1(t)$ is norm convergent, then for $t < t_0$ the remainder of solution series (23) can be made arbitrary small for sufficient large $n = n_0$ independently of ϵ . Then, using stated above asymptotic formulas, for each integer n every term of this series converges term by term to the limit operator $f_1(t)$ which is represented by series (17).

As stated above the mean field scaling limit (17) of solution (23) of the generalized quantum kinetic equation in the presence of initial correlations is governed by the quantum Vlasov-type kinetic equation with initial correlations (18).

Thus, we derived the quantum Vlasov-type kinetic equation with initial correlations (18) from the generalized quantum kinetic equation (28) in the mean field scaling limit. It is the same as the kinetic equation derived from the dual quantum Vlasov hierarchy (10) for the mean field limit marginal observables.

4.2 A mean field asymptotic behavior of marginal functionals of the state

As we noted above in Section 3 in case of initial data (14) the evolution of all possible correlations of quantum many-particle systems is described by marginal functionals of the state (24).

Since solution (23) of initial-value problem of the generalized quantum kinetic equation with initial correlations (28) converges to solution (17) of initial-value problem of the quantum Vlasov-type kinetic equation with initial correlations (18) as (29), and equalities (30) and (31) hold, then for a mean field asymptotic behavior of marginal functionals of the state (24) the following equalities are true:

$$\lim_{\epsilon \rightarrow 0} \left\| \epsilon^s F_s(t, 1, \dots, s \mid F_1(t)) - \prod_{j_1=1}^s \mathcal{G}_1^*(t, j_1) g_s(1, \dots, s) \prod_{j_2=1}^s \mathcal{G}_1^*(-t, j_2) \prod_{k=1}^s f_1(t, k) \right\|_{\mathcal{L}^1(\mathcal{H}_s)} = 0,$$

$$s \geq 2.$$

These equalities describe the propagation of initial correlations in time in the mean field scaling approximation.

5 Conclusion and outlook

In the paper the concept of quantum kinetic equations in case of the kinetic evolution, involving correlations of particle states at initial time, for instance, correlations characterizing the condensed states, was considered. Two approaches were developed with a view to this purpose. One approach based on the description of the evolution of quantum many-particle systems within the framework of marginal observables. Another method consists in the possibility in case of initial states specified by a one-particle marginal density operator and correlation operators to describe the evolution of states within the framework of a one-particle (marginal) density operator governed by the generalized quantum kinetic equation with initial correlations.

In case of pure states the quantum Vlasov-type kinetic equation with initial correlations (18) can be reduced to the Gross–Pitaevskii-type kinetic equation. Indeed, in this case the one-particle density operator $f_1(t) = |\psi_t\rangle\langle\psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t\rangle \in \mathcal{H}$ and its kernel has the following form: $f_1(t, q, q') = \psi(t, q)\psi^*(t, q')$. Then, if we consider quantum particles, interacting by the potential which kernel $\Phi(q) = \delta(q)$ is the Dirac measure, from kinetic equation (18) we derive the Gross–Pitaevskii-type kinetic equation

$$i\frac{\partial}{\partial t}\psi(t, q) = -\frac{1}{2}\Delta_q\psi(t, q) + \int dq' dq'' \mathbf{g}(t, q, q; q', q'')\psi(t, q'')\psi^*(t, q)\psi(t, q),$$

where the coupling ratio $\mathbf{g}(t, q, q; q', q'')$ of the collision integral is the kernel of the scattering length operator $\prod_{i_1=1}^2 \mathcal{G}_1^*(t, i_1) g_2(1, 2) \prod_{i_2=1}^2 \mathcal{G}_1^*(-t, i_2)$. If we consider a system of quantum particles without initial correlations, then this kinetic equation is the cubic nonlinear Schrödinger equation.

This paper deals with a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics. The obtained results can be extended to quantum systems of bosons or fermions [30].

We emphasize, that one of the advantages of the developed approach to the derivation of the quantum Vlasov-type kinetic equation with initial correlations from underlying dynamics governed by the generalized quantum kinetic equation with initial correlations enables to construct the higher-order corrections to the mean field evolution of quantum large particle systems.

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